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Quantization without gauge fixing: avoiding Gribov ambiguities through the physical projector

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Abstract. The quantization of gauge invariant systems usually proceeds through some gauge fixing procedure of one type or another. Typically for most cases, such gauge fixings are plagued by Gribov ambiguities, while it is only for an admissible gauge fixing that the correct dynamical description of the system is represented, especially with regards to nonperturbative phenomena. However, any gauge fixing procedure whatsoever may be avoided altogether, by using rather a recently proposed new approach based on the projection operator onto physical gauge invariant states only, which is necessarily free of any such issues. These different aspects of gauge invariant systems are explicitly analysed within a solvable $U(1)$ gauge invariant quantum mechanical model related to the dimensional reduction of Yang–Mills theory.

1. Introduction

As beautiful, elegant and powerful as is the general principle of local gauge invariance, there is also a price to be paid, especially when it comes to quantizing such theories. Indeed, the characteristic feature of systems possessing such symmetries is the presence among their degrees of freedom of redundant gauge variant variables required for a manifest realization of the gauge invariance principle, and possibly also of other symmetries such as space–time covariance. Consequently, the actual physical configuration space of these systems is typically quite intricate, whose nontrivial topology is at the heart of fundamental nonperturbative phenomena responsible for the rich physics implied by these theories.

The actual physical configuration space, being parametrized by the initial degrees of freedom modded out by the set of local (large and small) gauge transformations, corresponds to the set of gauge orbits in the configuration space of initial degrees of freedom. Usually the quantization of such systems implements first some gauge fixing procedure of one kind or another, which should in principle select among the initial degrees of freedom a single representative of each one of all the gauge orbits accessible to the system throughout its dynamical time evolution, thereby defining an admissible gauge fixing procedure. However, most gauge fixing procedures do not meet that requirement, and are then said to be plagued by

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a Gribov ambiguity [1,2]. In fact, two types of Gribov ambiguities ought to be distinguished†. The first type of Gribov problem, or 'local Gribov problem', arises when a gauge fixing procedure selects more than one representative of the same gauge orbit, the important point being however that the total number of these representatives must be summed by also accounting for the oriented integration measure induced on the physical configuration space of gauge orbits by the local integration measure on the initial configuration space [3,4]. The second type of possible Gribov problem, or 'global Gribov problem', arises when not all gauge orbits of the system are selected through some gauge fixing procedure [2]. Clearly, an admissible gauge fixing procedure is one which does not suffer either local or global Gribov problems.

In spite of the importance of the issue, especially when it comes to nonperturbative phenomena, the global Gribov problem is usually not specifically considered in the literature, although it is a possibility which often arises [2,3]. A local Gribov problem on the other hand, arises typically when the so-called Faddeev reduced phase space approach [5], for example, is developed towards the quantization of a gauge invariant system‡, even in the simplest cases [3]. The original example of a Gribov problem [1] could be of this type, but when accounting for the oriented integration measure over the space of gauge orbits, it may well be that the original Gribov example of gauge redundancy in a gauge fixing procedure is not a Gribov problem after all [4]. In fact, Gribov's suggestion for a resolution of local Gribov problems, by restriction to the fundamental domain of nonvanishing eigenvalues of the Faddeev–Popov determinant within the first Gribov horizon, has not met with the widest consensus. Indeed, the point has repeatedly been made [3,4] that one ought rather to count the multiple intersections of the gauge slice with the selected gauge orbits with an alternating signature determined by the oriented integration measure over the space of gauge orbits. Recently, that specific issue has been addressed again with the same conclusion in contradistinction to Gribov's suggestion, within the context of a solvable $U(1)$ gauge invariant quantum mechanical model inspired essentially by the dimensional reduction of ordinary Yang–Mills theories to $0 + 1$ dimensions [6].

Besides the Faddeev reduced phase space approach, there also exists the BFV-BRST invariant extended phase space formulation of gauge invariant systems [7], which in effect also imports the same issues of a gauge fixing procedure and the ensuing possible Gribov problems [3]. Even though these questions then arise in another disguise, nevertheless they need to be addressed specifically within that context as well, even in the simplest of examples [3]§.

The issue of Gribov problems thus needs to be considered on a case by case basis, not only for each physical system being studied, but also for each gauge fixing procedure which may be contemplated for a given system. Moreover, the Gribov problem issue is also very much dependent on the choice of boundary conditions which are being implemented for a given dynamical problem [1,2]. Hence, no procedure for resolving Gribov ambiguities in general, when they appear, can be formulated.

However, such issues arise specifically because of the apparent necessity of gauge fixing in gauge invariant systems. If gauge fixing could be avoided altogether, no Gribov problems would arise, and the issue would no longer need to be addressed. Indeed, the recent proposal [8] of the physical projector within Dirac's original approach [9] to the quantization of constrained

† For a detailed discussion and references to the original literature, see, for example, [3].

‡ Note that a *nonvanishing* Faddeev–Popov determinant establishes the absence of a local Gribov problem only but not necessarily of a global one, and then *only for infinitesimal gauge transformations* but not necessarily for finite (small and even large) ones. Thus, a nonvanishing Faddeev–Popov determinant does not necessarily define an admissible gauge fixing procedure in the sense considered above [3], contrary to the common usage of the word in the literature.

§ Note that when Gribov ambiguities arise for a given gauge fixing procedure, be it in the Faddeev reduced phase space approach or the BFV-BRST invariant one, they need to be addressed already at the classical level [3].

systems [3] precisely achieves [10] a correct integration over the space of gauge orbits of a system, in which all orbits are effectively included once and only once, without the necessity of performing a gauge fixing procedure of any kind. In particular, when considering the time evolution operator for gauge invariant states, the physical projector readily ensures that only physical states contribute as intermediate states to the physical propagator, a feature which is usually achieved only through gauge fixing to a reduced phase space description or by extending the quantum dynamics to include a ghost sector which compensates for those contributions from gauge variant states.

The purpose of this paper is to consider these different issues within the context of the simple $U(1)$ gauge invariant solvable model of [6], by relying on a series of considerations and results developed in [3, 11–13]. The definition of the model, which is very similar to the general class of systems already studied in [12] in terms of the physical projector, is recalled in section 2 together with its classical Hamiltonian formulation. Section 3 then discusses its Dirac quantization, by constructing the configuration space wavefunctions of the physical gauge invariant states. In section 4, a specific admissible gauge fixing procedure leading to a reduced phase space description of the model is developed, thereby enabling the explicit evaluation of the configuration space representation of the gauge invariant quantum evolution operator. Section 5 then considers the same matrix elements within the BFV-BRST approach, and illustrates how only admissible gauge fixing procedures—in the sense of the word as defined above—lead to the correct result for the gauge invariant evolution operator of physical states. All these results are then directly contrasted with those obtained in section 6 through the physical projector approach simply defined within the context of the Dirac quantization of the model, thereby again leading readily to the correct result for the physical propagator of the system to which only physical states contribute as intermediate states. Finally, section 7 presents the conclusions of the analysis.

2. The model and its classical Hamiltonian analysis

The dynamics of the $U(1)$ gauge invariant model of [6] is defined by the Lagrangian

$$L = \tfrac{1}{2}[(\dot{x} + g\xi y)^2 + (\dot{y} - g\xi x)^2 + (\dot{z} - \xi)^2] - V\left(\sqrt{x^2 + y^2}\right) \tag{1}$$

or in terms of polar coordinates

$$L = \tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}r^2(\dot{\theta} - g\xi)^2 + \tfrac{1}{2}(\dot{z} - \xi)^2 - V(r) \tag{2}$$

with, of course, $x = r\cos\theta$ and $y = r\sin\theta$. Here, $x(t)$, $y(t)$ and $z(t)$ are Cartesian coordinates, $\xi(t)$ is a gauge variable—essentially the time component of a $U(1)$ gauge connection after dimensional reduction to $0 + 1$ dimensions—and g is a gauge coupling constant. Henceforth, we also choose to work with the harmonic potential term

$$V(r) = \tfrac{1}{2}\omega^2 r^2. \tag{3}$$

It should be clear that this model possesses a $U(1)$ gauge invariance whereby the $(x(t), y(t))$ coordinates are rotated by some arbitrary time-dependent angle $\alpha(t)$ in the two-dimensional plane which they define, while at the same time the variables $z(t)$ and $\xi(t)$ are shifted by a quantity proportional either to $\alpha(t)$ or $\dot{\alpha}(t)$. Hence, this invariance of the system is best expressed in the polar parametrization, namely

$$r'(t) = r(t) \qquad \theta'(t) = \theta + \alpha(t) \qquad z'(t) = z(t) + \frac{1}{g}\alpha(t) \qquad \xi'(t) = \xi(t) + \frac{1}{g}\dot{\alpha}(t). \tag{4}$$

This property of the system already raises the issue of the choice of boundary conditions to be imposed on its degrees of freedom. Since later on we are interested in computing the quantum evolution operator of the model for gauge invariant states, boundary conditions need to be specified at two distinct moments in time t_i and t_f, with $t_f > t_i$. However, which of the degrees of freedom $(x, y, z; \xi)$ need to have their values specified at these values of t is a matter of their physical status.

In fact, two distinct physical interpretations of the degrees of freedom (x, y, z) are possible, among which two combinations are manifestly gauge invariant, namely

$$r(t) \qquad \varphi(t) = \theta(t) - gz(t) \tag{5}$$

while the third gauge *variant* combination may be chosen to be, for example,

$$\beta(t) = \theta(t) + gz(t). \tag{6}$$

In particular, note how the gauge invariant combinations r and φ show that the gauge orbits in the space (x, y, z) are nothing other than helicoidal curves of constant radius r, whose symmetry axis is parallel to the z-axis, and whose slope w.r.t. the (x, y) plane is set by the coupling g. Consequently, given any configuration $(r(t), \theta(t), z(t))$, a finite and unique gauge transformation of parameter function $\alpha(t)$ may always be found such that at all times $z'(t) = 0$, namely $\alpha(t) = -gz(t)$, so that also $\theta'(t) = \varphi(t)$.

Hence, one possible physical interpretation of the degrees of freedom of the system is to consider that its actual configuration space is the entire set of such helicoidal curves thus parametrized by the variables $0 \leqslant r < +\infty$ and $0 \leqslant \varphi < 2\pi$, and which are in one-to-one correspondence with the points $(r, \theta, z = 0)$ for all possible values of $0 \leqslant r < +\infty$ and $0 \leqslant \theta < 2\pi$. This is the point of view taken in [6] with regard to the nature of the configuration space of the system. Associated to the specific choice of admissible gauge fixing effected through the constraint $z(t) = 0$, which leaves no room for further nontrivial gauge transformations, the appropriate choice of boundary conditions is thus

$$\begin{aligned} r(t_i) &= r_i \qquad \theta(t_i) = \theta_i \qquad z(t_i) = 0 \\ r(t_f) &= r_f \qquad \theta(t_f) = \theta_f \qquad z(t_f) = 0 \end{aligned} \tag{7}$$

such that, in particular, $\varphi(t_{i,f}) = \theta_{i,f}$. Note that consistency with these boundary conditions requires the gauge transformation parameter $\alpha(t)$ to vanish at the end points, i.e. $\alpha(t_{i,f}) = 0$.

However, this situation suggests another possibility, in which the physical interpretation of the degrees of freedom is rather to consider that the actual configuration space of the system is indeed the set of all coordinates (x, y, z) in a three-dimensional Euclidean space, with the two sets of triplet values $(r(t_{i,f}), \theta(t_{i,f}), z(t_{i,f}))$ set to specific boundary values $(r_{i,f}, \theta_{i,f}, z_{i,f})$, knowing that any physical quantity computed with this choice of boundary conditions will depend only on the quantities $(r_{i,f}, \varphi_{i,f} = \theta_{i,f} - gz_{i,f})$. In this second interpretation, one decouples (one might say) the gauge transformations of the system for parameter functions $\alpha(t)$ which vanish at the endpoints in time $t_{i,f}$ from those which do not (a distinction which, within this interpretation, is consistent with the given boundary conditions), all in a continuous fashion. The latter transformations may be used to transform the boundary conditions such that $z(t)$ would vanish at the end points $t_{i,f}$. However, since we are solely interested in the propagator for physical states, the latter quantity may only depend on the gauge invariant combinations $r_{i,f}$ and $\varphi_{i,f}$ of the boundary values of trajectories. In other words, the physical propagator already only takes its values over the actual configuration space of gauge orbits of the first interpretation for the degrees of freedom of the model. Hence, in as far as the calculation of the physical propagator is concerned, the action of those gauge transformations for which $\alpha(t_{i,f}) \neq 0$ is already accounted for through the dependency on the variables $\varphi_{i,f}$, rather than $\theta_{i,f}$ and $z_{i,f}$ separately. Thus, given this second point of view, gauge equivalence classes of physical trajectories are characterized by having fixed end points at $t = t_{i,f}$, while all their other points $(x(t), y(t), z(t))$ associated to instants t distinct from $t_{i,f}$ are gauge transformed into one another with arbitrary parameter functions $\alpha(t)$ such that $\alpha(t_{i,f}) = 0$.

Apart from in section 4 where the specific gauge fixing $z(t) = 0$ is used from the outset, we shall thus consider the following choice of boundary conditions:

$$\begin{array}{lll} r(t_i) = r_i & \theta(t_i) = \theta_i & z(t_i) = z_i \\ r(t_f) = r_f & \theta(t_f) = \theta_f & z(t_f) = z_f \end{array} \tag{8}$$

to which the second interpretation of the degrees of freedom (x, y, z) is associated. Indeed, the structure of the Hamiltonian gauge invariance of the model then becomes very similar to that of the parametrized relativistic scalar particle [3], so that results established in the latter case may readily be borrowed for the calculation of the physical propagator in the present system, and for a discussion of Gribov problems in the context of its BFV-BRST invariant quantization.

The analysis of constraints of the system in its Hamiltonian formulation is straightforward enough. In its 'fundamental Hamiltonian description' (see [3] for the definition and justification of this concept), the model possesses three pairs of canonically conjugate phase space coordinates, (r, p_r), (θ, p_θ) and (z, p_z), subject to the first-class constraint $\phi = p_z + gp_\theta = 0$ of Lagrange multiplier ξ, and whose dynamics is generated by the first-class Hamiltonian

$$H = \frac{1}{2}p_r^2 + \frac{1}{2}\frac{1}{r^2}p_\theta^2 + \frac{1}{2}p_z^2 + \frac{1}{2}\omega^2 r^2 \qquad \{H, \phi\} = 0 \tag{9}$$

corresponding to the first-order Lagrangian

$$L_2 = \dot{r}p_r + \dot{\theta}p_\theta + \dot{z}p_z - \frac{1}{2}p_r^2 - \frac{1}{2}\frac{1}{r^2}p_\theta^2 - \frac{1}{2}p_z^2 - \frac{1}{2}\omega^2 r^2 - \xi[p_z + gp_\theta]. \tag{10}$$

In particular, the constraint ϕ is the generator of the $U(1)$ gauge symmetry on phase space through the symplectic structure defined by the Poisson brackets. Infinitesimal transformations may be exponentiated to finite ones, given by

$$\begin{array}{llll} r' = r & p'_r = p_r & \theta' = \theta + \alpha & p'_\theta = p_\theta \\ z' = z + \frac{1}{g}\alpha & p'_z = p_z & \xi' = \xi + \frac{1}{g}\dot{\alpha} & \end{array} \tag{11}$$

where $\alpha(t)$ is an arbitrary time-dependent function parametrizing Hamiltonian $U(1)$ gauge transformations, subject to the boundary conditions $\alpha(t_{i,f}) = 0$ when either of the choices of boundary conditions (7) or (8) applies.

Since the gauge transformations of the Lagrange multiplier ξ are independent of the phase space degrees of freedom, the general notion of Teichmüller space, namely the set of gauge orbits in the space of the Lagrange multipliers associated to all first-class constraints [3], applies in the present instance. Given the boundary conditions $\alpha(t_{i,f}) = 0$, it is clear that the variable

$$\gamma = \int_{t_i}^{t_f} \mathrm{d}t\, \xi(t) \tag{12}$$

defines a gauge invariant quantity in the space of Lagrange multiplier functions $\xi(t)$. Hence, γ is a coordinate parametrizing the Teichmüller space of the system, which in the present case is identified with the entire real line. Moreover, it should be quite clear that any gauge fixing in the space of Lagrange multiplier functions $\xi(t)$ induces a gauge fixing of the entire system itself in its phase space formulation, since we must have $\alpha(t_{i,f}) = 0$. As a matter of fact, it may easily be shown [3] that any admissible gauge fixing in the space $\xi(t)$, thus associated to a single covering of Teichmüller space in which all values of γ are obtained once and only once when accounting for the oriented integration measure on Teichmüller space, induces an admissible gauge fixing of the entire system itself. Thus, for example, a choice of gauge fixing in the space $\xi(t)$ such that the following set of functions is selected,

$$\xi(t;\gamma) = \frac{\gamma}{t_f - t_i} \tag{13}$$

where γ is a free parameter taking all possible real values once and only once, automatically induces an admissible gauge fixing of the system itself. Hence, for the present model, gauge fixing may be considered not only in terms of the dynamical degrees of freedom (x, y, z), as shown above through the choice $z(t) = 0$ for example, but it may also be effected through gauge fixing in the space of Lagrange multiplier functions $\xi(t)$. Note also how the characterization of these different gauge fixing procedures and the possible ensuing Gribov problems is strongly dependent on the choice of boundary conditions considered for the study of the time evolution of system configurations.

In the polar parametrization, the Hamiltonian equations of motion are†

$$\begin{aligned} &\dot r = p_r \qquad \dot p_r = \frac{p_\theta^2}{r^3} - \omega^2 r \qquad \dot\theta = \frac{p_\theta}{r^2} + g\xi \qquad \dot p_\theta = 0 \\ &\dot z = p_z + \xi \qquad \dot p_z = 0 \end{aligned} \tag{14}$$

subject further to the first-class constraint $\phi = p_z + gp_\theta = 0$. Note how, due to the *global* symmetries of the model under decoupled rotations in the (x, y) plane and translations along the z-axis, the angular momentum p_θ and the linear momentum p_z are each separately conserved quantities through time evolution, $p_\theta(t) = L$ and $p_z(t) = p$. However, the gauge invariance constraint $\phi = 0$ defining physical configurations, which generates time-dependent translations in z coupled to time-dependent rotations in (x, y), brings into further relation these two conserved quantities, such that $p + gL = 0$.

Given the choice of boundary conditions (8), the construction of the general solution to these equations of motion proceeds as follows. First, the values for the rotational energy‡ E_r and for the angular momentum L must be determined such that§

$$t_f - t_i = \int_{r_i}^{r_f} \mathrm{d}u \frac{\pm 1}{\sqrt{2[E_r - V(u) - \frac{L^2}{2u^2}]}} \qquad L = \frac{\varphi_f - \varphi_i}{g^2(t_f - t_i) + \int_{t_i}^{t_f} \frac{\mathrm{d}t}{r^2(t)}} \tag{15}$$

where the solution for $r(t)$ is implicitly defined by the integral

$$t - t_i = \int_{r_i}^{r(t)} \mathrm{d}u \frac{\pm 1}{\sqrt{2[E_r - V(u) - \frac{L^2}{2u^2}]}}. \tag{16}$$

In both this latter expression as well as in the previous one particularized to $t = t_f$, the ± 1 sign under the integral stands for the sign of the time derivative $\mathrm{d}r(t)/\mathrm{d}t$. Once the values for E_r and L thereby determined, the remaining phase space variables are given by

$$\begin{aligned} &p_r(t) = \dot r(t) \\ &\theta(t) = \theta_i + L\int_{t_i}^{t} \mathrm{d}t' \frac{1}{r^2(t')} + g\int_{t_i}^{t} \mathrm{d}t'\,\xi(t') \qquad p_\theta(t) = L \\ &z(t) = z_i - gL(t - t_i) + \int_{t_i}^{t} \mathrm{d}t'\,\xi(t') \qquad p_z(t) = p = -gL. \end{aligned} \tag{17}$$

† Incidentally, given the first-order Lagrangian (10), the Hamiltonian reduction whereby all conjugate phase space coordinates p_r, p_θ and p_z are solved for through the Hamiltonian equations of motion, leads back precisely to the original Lagrangian formulation (2) of the model. The same would have applied, of course, had one worked in terms of Cartesian coordinates.

‡ The total energy E of the system is then given by $E = E_r + p^2/2$.

§ With our choice of harmonic potential $V(r) = \omega^2 r^2/2$, these conditions may be solved explicitly, but the ensuing expressions are not very informative, and are thus not given.

In view of the gauge transformation of the Lagrange multiplier in (11), note how the function $\xi(t)$ appearing in these solutions parametrizes the gauge freedom, i.e. the gauge redundancy, of the general solutions to the equations of motion of the original Euler–Lagrange equations associated to the Lagrangian (2). In particular, the gauge invariant combination $\varphi(t) = \theta(t) - gz(t)$, given by

$$\varphi(t) = [\theta_i - gz_i] + L\int_{t_i}^{t} \mathrm{d}t' \left(\frac{1}{r^2(t')} + g^2\right) \tag{18}$$

is indeed independent of the Lagrange multiplier function $\xi(t)$.

However, for consistency of the construction of the solution, the choice of Lagrange multiplier $\xi(t)$ must be such that the associated Teichmüller parameter γ takes the value

$$\gamma = \int_{t_i}^{t_f} \mathrm{d}t\, \xi(t) = (z_f - z_i) + gL(t_f - t_i). \tag{19}$$

Hence, if a gauge fixing procedure is implemented such that the Teichmüller parameter values γ of the Lagrange multipliers $\xi(t)$ thereby effectively selected do not include this specific value required by the choice of boundary conditions, the set of gauge orbits of the system retained through gauge fixing does not include the specific one which solves the equations of motion with this choice of boundary conditions. Clearly, for an admissible gauge fixing, such a situation never arises since all possible values of γ are then included once and only once. This simple remark thus illustrates, at the classical level already, how a nonadmissible gauge fixing procedure, thus suffering either a local or a global Gribov problem, excludes from the retained configurations of the system or includes with too large a multiplicity certain subsets which physically are perfectly acceptable and should thus remain accessible to the system throughout its dynamical evolution with a single multiplicity for each of its gauge orbits.

The set of Teichmüller parameter values selected through the admissible gauge fixing choice $z(t) = 0$ will be discussed in detail in section 4. Generally, one-parameter sets of functions $\xi(t)$ with their associated values for the Teichmüller parameter γ may be obtained through any gauge fixing procedure leading to an equation of the form [3, 14, 15]

$$\dot{\xi} = F(\xi) \tag{20}$$

where $F(\xi)$ is some given function. Indeed, the solution $\xi(t; \xi_f)$ to this condition is determined in terms of a single integration constant, which for later convenience we choose to be the value taken by $\xi(t; \xi_f)$ at $t = t_f$, namely $\xi_f = \xi(t_f; \xi_f)$. Correspondingly, as the value of ξ_f runs from $-\infty$ to $+\infty$, the Teichmüller parameter $\gamma(\xi_f) = \int_{t_i}^{t_f} \mathrm{d}t\, \xi(t; \xi_f)$ takes its values in a certain domain $\mathcal{D}_\gamma[F]$ and with a certain covering of that domain which both directly depend on the function $F(\xi)$ in (20). Within this framework, an admissible gauge fixing is related to a function $F(\xi)$ such that the domain $\mathcal{D}_\gamma[F]$ is the entire real line and with a covering such that all values for γ are obtained once and only once.

Since the respective Teichmüller spaces are identical, let us consider some of the specific examples discussed in the case of the parametrized relativistic scalar particle in [3, 15]. The choice

$$F(\xi) = a\xi + b \tag{21}$$

implies

$$\begin{aligned} \xi(t; \xi_f) &= \left(\xi_f + \frac{b}{a}\right) \mathrm{e}^{a(t-t_f)} - \frac{b}{a} \\ \gamma(\xi_f) &= \frac{1}{a}\left(\xi_f + \frac{b}{a}\right)(1 - \mathrm{e}^{-a(t_f-t_i)}) - \frac{b}{a}(t_f - t_i) \end{aligned} \tag{22}$$

which clearly thus defines an admissible choice of gauge fixing, since as ξ_f varies from $-\infty$ to $+\infty$ the corresponding domain $\mathcal{D}_\gamma[F]$ is then indeed the entire real line with each value of γ obtained once and only once, irrespective of the values of the arbitrary parameters a and b defining the function $F(\xi)$. In particular, the simplest choice of such an admissible gauge fixing is obtained with $F(\xi) = 0$ [14], thereby leading to the set of Lagrange multiplier functions $\xi(t)$ mentioned in (13) with $\gamma(\xi_f) = \xi_f(t_f - t_i)$.

A choice of a quadratic function for $F(\xi)$, however, is associated to a nonadmissible gauge fixing. Indeed [3], even though all real values of the Teichmüller parameter are then obtained, they are obtained twice while ξ_f runs from $-\infty$ to $+\infty$, *and with opposite orientations*. Hence, effectively, the actual covering of Teichmüller space which is implied by such a choice for $F(\xi)$ vanishes, establishing the nonadmissibility of such a gauge fixing, which thus possesses a global Gribov problem, no gauge orbit being effectively retained. Nevertheless, as the coefficient of the quadratic term in $F(\xi)$ vanishes, the previous class of admissible gauge fixings is recovered [3].

The only other example borrowed from [3] we shall mention here is

$$F(\xi) = a\xi^3 \qquad a > 0. \tag{23}$$

Correspondingly, one finds

$$\xi(t;\xi_f) = \frac{\xi_f}{\sqrt{1 + 2a\xi_f^2(t - t_f)}} \qquad \gamma(\xi_f) = \frac{1}{a\xi_f}\left[\sqrt{1 + 2a\xi_f^2(t_f - t_i)} - 1\right]. \tag{24}$$

Consequently, for all nonvanishing positive values of the parameter a, this choice leads to a nonadmissible gauge fixing of the system. Indeed, as ξ_f runs from $-\infty$ to $+\infty$, the associated domain in Teichmüller space reduces to the finite interval $\mathcal{D}_\gamma[F] = [-\sqrt{2(t_f - t_i)/a}, \sqrt{2(t_f - t_i)/a}]$ with a single covering. In other words, even though it does not suffer a local Gribov problem, this gauge fixing is nonadmissible since it suffers a global one. Nevertheless, in the limit where the parameter a vanishes, the admissible gauge fixing implied by $F(\xi) = 0$ is indeed recovered.

Incidentally, note that even already at the classical level, these simple examples illustrate the general fact that the gauge invariant physics described by gauge invariant systems *is not independent* of the gauge fixing procedure to which they are subjected [3,15], contrary to what seems to be generally believed to be true. Gauge invariance of physical quantities is not all there is to gauge invariant systems; it is also imperative that the actual space of gauge orbits of the system be properly accounted for by any description based on some gauge fixing procedure. This can only be achieved through an admissible gauge fixing, even though any gauge fixing procedure, *including those suffering local or global Gribov problems*, always leads to gauge invariant results for physical observables.

3. Dirac quantization

Dirac's quantization of the model simply consists in the canonical quantization of the previous Hamiltonian formulation of the system, with the constraint of gauge invariance $\phi = 0$ imposed at the operatorial level in order to define physical, i.e. gauge invariant, states. For convenience, we choose to work in the polar parametrization (r, θ, z) of the degrees of freedom of the system, which thus requires the construction of representations of the associated Heisenberg algebra in these curvilinear coordinates parametrizing the three-dimensional Euclidean space defined by the Cartesian coordinates (x, y, z). Moreover, given our intent of computing the propagator of physical states, we shall consider from the outset the configuration space representation of the canonical commutation relations in polar coordinates.

For this purpose, we rely on the classification [13,16] of representations of the Heisenberg algebra in arbitrary coordinate systems set up on arbitrary manifolds. In the present instance, since the Euclidean space (x, y, z) is simply connected, only the trivial representation of the Heisenberg algebra exists, while the differential operator representation of the conjugate momenta operators $\hat{p}_r$, $\hat{p}_\theta$ and $\hat{p}_z$ is determined by the metric structure of this configuration space, expressed in polar coordinates by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + \mathrm{d}z^2. \tag{25}$$

Consequently [13], the configuration space wavefunction inner product is defined by the integration measure

$$\int_0^{+\infty} \mathrm{d}r\, r \int_0^{2\pi} \mathrm{d}\theta \int_{-\infty}^{+\infty} \mathrm{d}z \tag{26}$$

while the conjugate momenta operators are represented by the differential operators

$$\hat{p}_r : -\frac{\mathrm{i}\hbar}{\sqrt{r}}\partial_r\sqrt{r} \qquad \hat{p}_\theta : -\mathrm{i}\hbar\partial_\theta \qquad \hat{p}_z : -\mathrm{i}\hbar\partial_z \tag{27}$$

the coordinate operators $\hat{r}$, $\hat{\theta}$ and $\hat{z}$ being, of course, represented on configuration space wavefunctions through simple multiplication by the associated eigenvalues r, θ and z.

This does not yet specify the choice of quantum Hamiltonian for the system, in correspondence with the classical first-class Hamiltonian H in (9). However, the canonical choice corresponding to the usual scalar Laplacian differential operator, is defined by [13]

$$\hat{H} = \frac{1}{2}\frac{1}{\sqrt{\hat{r}}}\hat{p}_r\hat{r}\hat{p}_r\frac{1}{\sqrt{\hat{r}}} + \frac{1}{2}\frac{1}{\hat{r}^2}\hat{p}_\theta^2 + \frac{1}{2}\hat{p}_z^2 + \frac{1}{2}\omega^2\hat{r}^2 \tag{28}$$

which, in terms of the above representation of the conjugate momenta operators, also reads

$$\hat{H} : -\frac{\hbar^2}{2}\left[\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2\right] + \frac{1}{2}\omega^2 r^2. \tag{29}$$

Finally, the gauge generator operator $\hat{\phi} = \hat{p}_z + g\hat{p}_\theta$ is thus also represented by

$$\hat{\phi} : -\mathrm{i}\hbar(\partial_z + g\partial_\theta). \tag{30}$$

In particular, by definition, the configuration space wavefunctions of gauge invariant states are annihilated by this latter differential operator.

Let us first consider the diagonalization of the first-class Hamiltonian $\hat{H}$, whose set of eigenstates thus defines a basis for the full space of quantum states of the system, of which a specific linear subspace is that of the gauge invariant physical states annihilated by the gauge generator $\hat{\phi}$. In the same manner as at the classical level, since $\hat{H}$, $\hat{p}_\theta$ and $\hat{p}_z$ are all commuting operators, a common diagonalization of all these three operators may be found, thereby also diagonalizing the gauge generator $\hat{\phi}$. Thus, a basis for the physical states of the system is defined by the specific subset of this diagonalizing basis of states whose $\hat{\phi}$ eigenvalue vanishes identically.

With these considerations in mind, the resolution of the Schrödinger equation

$$\left\{-\frac{\hbar^2}{2}\left[\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2\right] + \frac{1}{2}\omega^2 r^2\right\}\psi(r,\theta,z) = E\psi(r,\theta,z) \tag{31}$$

is straightforward enough. The eigenvalue spectrum is given by

$$E_{m,\ell,p} = \tfrac{1}{2}p^2 + \hbar\omega[2m + |\ell| + 1] \tag{32}$$

with $-\infty < p < +\infty$, $m = 0, 1, 2, \ldots$ and $\ell = 0, \pm 1, \pm 2, \ldots$, while the corresponding eigenstate configuration space wavefunctions are

$$\begin{aligned}\psi_{m,\ell,p}(r,\theta,z) &= \langle r,\theta,z|m,\ell,p\rangle \\ &= (-1)^m \left(\frac{\omega}{\pi\hbar}\right)^{1/2} \left(\frac{1}{2\pi\hbar}\right)^{1/2} \left(\frac{m!}{(m+|\ell|)!}\right)^{1/2} \mathrm{e}^{\mathrm{i}\ell\theta}\mathrm{e}^{\mathrm{i}pz/\hbar} u^{|\ell|} \mathrm{e}^{-u^2/2} L_m^{|\ell|}(u^2) \qquad (33)\end{aligned}$$

where $L_m^{|\ell|}(x)$ are the usual Laguerre polynomials, the variable u is defined by $u = r\sqrt{\omega/\hbar}$ and the choice of overall phase $(-1)^m$ is made for later convenience. The normalization of these states is such that

$$\langle m,\ell,p|m',\ell',p'\rangle = \delta_{mm'}\delta_{\ell\ell'}\delta(p-p'). \qquad (34)$$

In particular, a basis for the physical states of the model is provided by these wavefunctions with the further restriction that their $\hat{\phi}$ eigenvalue vanishes, namely

$$p = -\hbar g\ell. \qquad (35)$$

Note that the dependency of the associated wavefunctions on the gauge variant variables θ and z combines into a single phase dependency on the specific gauge invariant combination $\varphi = \theta - gz$, namely with the following recombination of phase factors:

$$\mathrm{e}^{\mathrm{i}\ell\theta}\mathrm{e}^{\mathrm{i}pz/\hbar} \to \mathrm{e}^{\mathrm{i}\ell(\theta-gz)} = \mathrm{e}^{\mathrm{i}\ell\varphi} \qquad (36)$$

as was indeed expected.

This explicit resolution of the quantized model may of course also be achieved through algebraic methods [6,12]. For that purpose, it is more appropriate to first consider the Cartesian parametrization of its degrees of freedom (x, y, z), and the definition of the usual creation and annihilation operators (only the latter are recalled here),

$$a_1 = \sqrt{\frac{\omega}{2\hbar}}\left[\hat{x} + \frac{\mathrm{i}}{\omega}\hat{p}_x\right] \qquad a_2 = \sqrt{\frac{\omega}{2\hbar}}\left[\hat{y} + \frac{\mathrm{i}}{\omega}\hat{p}_y\right]. \qquad (37)$$

However, rotational invariance of the model in the (x, y) plane calls for the introduction of an helicity-like basis of creation and annihilation operators, defined by [12]

$$a_\pm = \frac{1}{\sqrt{2}}[a_1 \mp \mathrm{i}a_2] \qquad a_\pm^\dagger = \frac{1}{\sqrt{2}}[a_1^\dagger \pm \mathrm{i}a_2^\dagger]. \qquad (38)$$

In term of these quantities, the first-class Hamiltonian and the gauge generator read

$$\hat{H} = \tfrac{1}{2}\hat{p}_z^2 + \hbar\omega[a_+^\dagger a_+ + a_-^\dagger a_-] \qquad \hat{\phi} = \hat{p}_z + \hbar g[a_+^\dagger a_+ - a_-^\dagger a_-]. \qquad (39)$$

Hence, the orthonormalized helicity Fock state basis, extended here to $\hat{p}_z$ momentum eigenstates and defined by

$$|n_\pm, p\rangle = \frac{1}{\sqrt{n_+!n_-!}}(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}|0, p\rangle \qquad (40)$$

and with the Fock vacuum $|0, p\rangle$ being such that

$$a_\pm|0,p\rangle = 0 \qquad \hat{p}_z|0,p\rangle = p|0,p\rangle \qquad \langle 0,p|0,p'\rangle = \delta(p-p') \qquad (41)$$

provides a direct diagonalization both of the Hamiltonian $\hat{H}$ and of the gauge generator $\hat{\phi}$. The energy spectrum is thus

$$\hat{H}|n_\pm, p\rangle = E_{n_\pm,p}|n_\pm, p\rangle \qquad E_{n_\pm,p} = \tfrac{1}{2}p^2 + \hbar\omega(n_+ + n_- + 1) \qquad (42)$$

while the physical state condition $\hat{\phi} = 0$ leads to the restriction

$$p = -\hbar g(n_+ - n_-). \qquad (43)$$

To establish complete identity with the previous results, one now only needs to determine the configuration space wavefunction representation of the states $|n_\pm, p\rangle$. This is a straightforward exercise using the differential operator representations of the operators $\hat{p}_r$ and $\hat{p}_\theta$ introduced previously. One then finds [6] precisely the same wavefunctions as those given in (33), with the following correspondence:

$$\langle r, \theta, z|n_\pm, p\rangle = \psi_{m,\ell,p}(r, \theta, z) \qquad m = \min(n_+, n_-) \qquad \ell = n_+ - n_-. \tag{44}$$

Since the first-class Hamiltonian commutes with the gauge generator, any physical state retains this quality under time evolution generated by the evolution operator of the system given by the time-ordered expression

$$\hat{U}(t_f, t_i) = T\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f}\mathrm{d}t'[\hat{H}+\xi(t')\hat{\phi}]} \tag{45}$$

where $\xi(t)$ stands for an arbitrary choice of Lagrange multiplier. Nevertheless, this fact does not allow a direct evaluation of the configuration space matrix elements of the evolution operator to which only physical states would contribute as intermediate states, since configuration space eigenstates $|r, \theta, z\rangle$ do not define gauge invariant states. Hence, it appears that in order to evaluate the configuration space propagator of the system restricted to gauge invariant states only, one needs first to implement some gauge fixing procedure by which the contributions from all gauge *variant* variables are removed, while retaining only those of the physical states. The calculation of this physical state propagator through gauge fixing is the purpose of the next two sections, before showing in section 6 how the same goal may readily be reached simply by using the physical projector [8] within Dirac's quantization of the system, which requires no gauge fixing whatsoever.

4. Reduced phase space quantization

In this section, we consider Faddeev's reduced phase space formulation [3, 5] of the model given the gauge fixing condition $z(t) = 0$, to which the choice of boundary conditions (7) thus applies. As was discussed previously, we know that this choice of gauge fixing is admissible†. Hence, the canonical quantization of the corresponding Hamiltonian formulation should allow for the calculation of the physical propagator of the system, to which all gauge orbits of the model contribute once and only once.

Together with the first-class generator $\phi = p_z + gp_\theta$ of the $U(1)$ gauge symmetry, the gauge fixing condition

$$\Omega = z = 0 \tag{46}$$

defines a set of second-class constraints, whose Faddeev–Popov determinant does not vanish,

$$\{\phi, \Omega\} = -1. \tag{47}$$

In particular, the requirement that the gauge fixing condition $\Omega = 0$ be maintained under time evolution generated by the total Hamiltonian $H_T = H + \xi\phi$, namely $\dot{\Omega} = 0$, implies the following specification of the Lagrange multiplier:

$$\xi = -p_z = gp_\theta. \tag{48}$$

Hence, given the fact that $p_\theta(t)$ keeps a constant value L for solutions to the equations of motion, the associated Teichmüller parameter

$$\gamma = \int_{t_i}^{t_f} \mathrm{d}t\, \xi(t) = gL(t_f - t_i) \tag{49}$$

† Other reduced phase space gauge fixings, such as $z(t) - \lambda x(t) = 0$ with λ a fixed positive parameter, are discussed in [6].

does indeed obey the constraint (19) set by the boundary conditions (7). Of course, the fact that this constraint is met is a consequence of the admissibility of the gauge fixing condition $z(t) = 0$.

The reduction of the system, and in particular the decoupling of the (z, p_z) sector of phase space degrees of freedom through the relations

$$z = 0 \qquad p_z = -gp_\theta \tag{50}$$

necessitates the calculation of the Dirac brackets associated to the second-class constraints (ϕ, Ω). One easily finds that the Dirac brackets among the remaining phase space conjugate pairs (r, p_r) and (θ, p_θ) are identical to their original Poisson brackets. In addition, the reduced Hamiltonian appropriate to this reduced phase space formulation of the model is given by

$$H_{\text{red}} = \frac{1}{2}p_r^2 + \frac{1}{2}\left[\frac{1}{r^2} + g^2\right]p_\theta^2 + \frac{1}{2}\omega^2 r^2. \tag{51}$$

In particular, the corresponding equations of motion,

$$\dot{r} = p_r \qquad \dot{p}_r = \frac{1}{r^3}p_\theta^2 - \omega^2 r \qquad \dot{\theta} = \left(\frac{1}{r^2} + g^2\right)p_\theta \qquad \dot{p}_\theta = 0 \tag{52}$$

coincide with those obtained from the original equations of motion (14) in which the relations $z = 0$, $p_z = -gp_\theta$ and $\xi = -p_z = gp_\theta$ are substituted.

The canonical quantization of this formulation of the model in polar coordinates is straightforward enough, since it coincides with that of the (r, θ) sector in Dirac's quantization. Hence, we may immediately transcribe the corresponding configuration space representations of quantum operators. In particular, a basis of the quantum space of states, which are now necessarily all gauge invariant ones, is provided by those states which diagonalize both the angular momentum operator $\hat{p}_\theta = -\mathrm{i}\hbar\partial_\theta$ and the Schrödinger operator associated to the reduced Hamiltonian above, namely

$$\left\{-\frac{\hbar^2}{2}\left[\partial_r^2 + \frac{1}{r}\partial_r + \left(\frac{1}{r^2} + g^2\right)\partial_\theta^2\right] + \frac{1}{2}\omega^2 r^2\right\}\psi(r, \theta) = E\psi(r, \theta). \tag{53}$$

The explicit resolution of this equation shows that the eigenvalue spectrum is given by

$$E_{m,\ell} = \tfrac{1}{2}\hbar^2 g^2 \ell^2 + \hbar\omega[2m + |\ell| + 1] \tag{54}$$

with $m = 0, 1, 2, \ldots$ and $\ell = 0, \pm 1, \pm 2, \ldots$, while the corresponding eigenstate configuration space wavefunctions are

$$\psi_{m,\ell}(r, \theta) = \langle r, \theta | m, \ell\rangle = (-1)^m \left(\frac{\omega}{\pi\hbar}\right)^{1/2} \left(\frac{m!}{(m + |\ell|)!}\right)^{1/2} \mathrm{e}^{\mathrm{i}\ell\theta} u^{|\ell|} \mathrm{e}^{-u^2/2} L_m^{|\ell|}(u^2) \tag{55}$$

with the same notations as in (33) and a normalization such that

$$\langle m, \ell | m', \ell'\rangle = \delta_{mm'}\delta_{\ell\ell'}. \tag{56}$$

Except for the normalization factor $1/(2\pi\hbar)^{1/2}$ stemming from the (z, p_z) sector which has been reduced in the present approach, this energy spectrum as well as these wavefunctions coincide with those established for physical states in Dirac's quantization (recall that we have here $z = 0$ so that $\varphi = \theta - gz = \theta$).

The calculation of the physical propagator is now immediate. Since it is defined by the matrix elements

$$P_{\text{red}}(i \to f) = \langle r_f, \theta_f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(t_f - t_i)\hat{H}_{\text{red}}} | r_i, \theta_i\rangle, \tag{57}$$

we explicitly have

$$P_{\text{red}}(i \to f) = \sum_{m=0}^{+\infty} \sum_{\ell=-\infty}^{+\infty} \langle r_f, \theta_f | m, \ell\rangle \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(t_f - t_i)E_{m,\ell}} \langle m, \ell | r_i, \theta_i\rangle. \tag{58}$$

Given the wavefunctions established above, the summation over the integer m is possible in terms of Bessel functions of the first kind, leading finally to the following expression for the configuration space propagator of physical states†,

$$P_{\mathrm{red}}(i \to f) = \frac{\omega}{2\mathrm{i}\pi\hbar \sin\omega\Delta t} \mathrm{e}^{\frac{\mathrm{i}}{2}\frac{\cos\omega\Delta t}{\sin\omega\Delta t}(u_f^2+u_i^2)} \times \sum_{\ell=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}|\ell|} \mathrm{e}^{-\frac{\mathrm{i}}{2}\hbar\Delta t g^2 \ell^2} \mathrm{e}^{\mathrm{i}\ell(\theta_f-\theta_i)} J_{|\ell|}\left(\frac{u_f u_i}{\sin\omega\Delta t}\right) \tag{59}$$

where $\Delta t = t_f - t_i$. This latter result will thus serve as the point of comparison for the physical propagator obtained through the other two quantization approaches considered in this paper, namely the so-called BFV-BRST invariant formulation of gauge invariant systems and the physical projector construction of [8].

5. BFV-BRST quantization

As opposed to the reduced phase space approach, the BFV-BRST one [3, 7] extends the set of dynamical degrees of freedom in the following manner. First, the Lagrange multiplier $\xi(t)$ is promoted to being a dynamical variable by introducing its conjugate momentum $p_\xi(t)$, thereby leading now to two first-class constraints

$$G_{a=1} = p_\xi = 0 \qquad G_{a=2} = \phi = p_z + g p_\theta = 0. \tag{60}$$

Note that we still have

$$\{G_a, G_b\} = 0 \qquad \{H, G_a\} = 0 \qquad a, b = 1, 2. \tag{61}$$

Next, in order to compensate for these additional degrees of freedom, further dynamical variables of opposite Grassmann parity are introduced, the BFV ghosts $\eta^a(t)$ and $\mathcal{P}_a(t)$, $(a = 1, 2)$, each such pair being canonically conjugate variables with graded Poisson brackets

$$\{\eta^a, \mathcal{P}_b\} = -\delta^a_b \qquad a, b = 1, 2. \tag{62}$$

In addition, the BFV ghosts have the following properties under complex conjugation: $(\eta^a)^* = \eta^a$ and $(\mathcal{P}_a)^* = -\mathcal{P}_a$ $(a = 1, 2)$.

Within this framework, the original Hamiltonian gauge invariance generated by the first-class constraint $\phi = G_2$ is now traded for a global BRST symmetry generated by the Grassmann odd BRST charge Q_B, which for the present model is given by

$$Q_B = \eta^a G_a = \eta^1 p_\xi + \eta^2 [p_z + g p_\theta] \tag{63}$$

and is characterized by the nilpotency property $\{Q_B, Q_B\} = 0$ as well as being real under complex conjugation, $(Q_B)^* = Q_B$.

Similarly, the original first-class Hamiltonian H may also be extended to a BRST invariant one, H_B, which in the present case is identical to H, $H_B = H$. However, in the same way that the time evolution of the system in Dirac's construction is generated by the Hamiltonian H to which an arbitrary linear combination of the first-class constraints is added, here as well time evolution in this extended phase space description of the system is generated by the most general possible BRST invariant Hamiltonian based on H_B, given by $H_{\mathrm{eff}} = H_B - \{\Psi, Q_B\}$, Ψ being *a priori* a totally arbitrary function on the extended phase space, of Grassmann odd parity and odd under complex conjugation, while the nilpotency property of the BRST charge ensures that $\{H_{\mathrm{eff}}, Q_B\} = 0$.

† Had the factor $\exp(-\mathrm{i}\hbar\Delta t g^2\ell^2/2)$ been absent from this expression, the summation over ℓ would have been possible as well [13], leading, of course, to the usual propagator for the two-dimensional spherically symmetric harmonic oscillator of angular frequency ω and unit mass $m = 1$.

Given any choice for the function Ψ, the equations of motion of the system within this extended framework are easily established. In order to construct gauge invariant solutions however, it is imperative to impose BRST invariant boundary conditions which extend the choice considered previously in (8). A general discussion [3] shows that the following conditions always meet this requirement of BRST invariance:

$$\begin{array}{lll} p_\xi(t_i) = 0 & \mathcal{P}_1(t_i) = 0 & \eta^2(t_i) = 0 \\ p_\xi(t_f) = 0 & \mathcal{P}_1(t_f) = 0 & \eta^2(t_f) = 0 \end{array} \tag{64}$$

a fact which in the present case is explicitly confirmed by considering the BRST transformations of the extended phase space variables,

$$\begin{array}{llll} \delta_B r = \{r, Q_B\} = 0 & \delta_B p_r = 0 & \delta_B \theta = g\eta^2 & \delta_B p_\theta = 0 \\ \delta_B z = \eta^2 & \delta_B p_z = 0 & \delta_B \xi = \eta^1 & \delta_B p_\xi = 0 \\ \delta_B \eta^1 = 0 & \delta_B \mathcal{P}_1 = -p_\xi & \delta_B \eta^2 = 0 & \delta_B \mathcal{P}_2 = -[p_z + gp_\theta]. \end{array} \tag{65}$$

In the remainder of this section, we shall consider the specific choice [3, 15]

$$\Psi = F(\xi)\mathcal{P}_1 + \xi\mathcal{P}_2 \tag{66}$$

where $F(\xi)$ is an arbitrary function. The corresponding effective Hamiltonian is then

$$H_{\mathrm{eff}} = \frac{1}{2}p_r^2 + \frac{1}{2r^2}p_\theta^2 + \frac{1}{2}p_z^2 + \frac{1}{2}\omega^2 r^2 + \xi[p_z + gp_\theta] + F(\xi)p_\xi - F'(\xi)\mathcal{P}_1\eta^1 - \mathcal{P}_2\eta^1 \tag{67}$$

from which it is possible to derive the equations of motion for all the extended phase space variables (r, p_r), (θ, p_θ), (z, p_z), (ξ, p_ξ) and $(\eta^a, \mathcal{P}_a)$. Among these equations, the one for the Lagrange multiplier is simply

$$\dot{\xi} = F(\xi). \tag{68}$$

In other words, within the BFV-BRST framework, the function Ψ provides the 'gauge fixing fermion function', which for the choice in (66) implies a gauge fixing of the system in its space of Lagrange multiplier functions $\xi(t)$ precisely of the type discussed in (20), for which admissible and nonadmissible examples were described.

Rather than now considering the construction of the general solution to these equations of motion at the classical level, let us turn immediately to the BRST quantization of the system. Through the redefinitions $c^a = \hat{\eta}^a$ and $b_a = \frac{\mathrm{i}}{\hbar}\hat{\mathcal{P}}_a$ $(a = 1, 2)$, the operator algebra in the ghost sector is specified by the set of anticommutation relations

$$\begin{array}{lll} \{c^a, b_b\} = \delta^a_b & (c^a)^2 = 0 & (b_a)^2 = 0 \\ c^{a\dagger} = c^a & b_a^\dagger = b_a & a, b = 1, 2 \end{array} \tag{69}$$

thus defining the usual so-called (b, c) ghost system whose representation theory is straightforward and well known [3]. Further, the definitions of the BRST and gauge fixing fermion operators, $\hat{Q}_B$ and $\hat{\Psi}$, read as those at the classical level above, while the BRST invariant effective quantum Hamiltonian operator $\hat{H}_{\mathrm{eff}}$ is now given by

$$\hat{H}_{\mathrm{eff}} = \hat{H} + \frac{\mathrm{i}}{\hbar}\{\hat{\Psi}, \hat{Q}_B\} \tag{70}$$

where the choice of normal ordering in the p_r^2 contribution to the first-class Hamiltonian operator $\hat{H}$ is that of section 3. Explicitly, one finds

$$\hat{H}_{\mathrm{eff}} = \frac{1}{2}\hat{p}_r^2 - \frac{\hbar^2}{8\hat{r}^2} + \frac{1}{2\hat{r}^2}\hat{p}_\theta^2 + \frac{1}{2}\hat{p}_z^2 + \frac{1}{2}\omega^2\hat{r}^2 + \hat{\xi}[\hat{p}_z + g\hat{p}_\theta] + F(\hat{\xi})\hat{p}_\xi - \mathrm{i}\hbar F'(\hat{\xi})c^1 b_1 - \mathrm{i}\hbar c^1 b_2. \tag{71}$$

We are now in a position to compute the physical propagator of the system, namely the matrix elements of the BRST invariant evolution operator

$$\hat{U}_{\mathrm{BRST}}(t_f, t_i) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Delta t \hat{H}_{\mathrm{eff}}} \tag{72}$$

for the BRST invariant external states of the quantized system which correspond to the choice of BRST invariant boundary conditions in (8) and (64). However, since the extended sectors of variables (ξ, p_ξ) and $(\eta^a, \mathcal{P}_a)$ as well as the structure of the Hamiltonian gauge algebras of the present model and of the parametrized relativistic scalar particle are identical, only the outline of the calculation will be discussed here, whose complete details are thoroughly presented in [3] for the latter system. In particular, for reasons presented in that work, the calculation of the matrix elements we are interested in is best performed using the path integral representation over the BFV extended phase space. The construction of a discretized but exact expression for that path integral proceeds in the usual fashion, by considering the $(N-1)$-times insertion of the spectral decomposition of the identity operator on the space of quantum states between the N factors appearing in the following rewriting of the evolution operator, $\hat{U}_{\mathrm{BRST}}(t_f, t_i) = [\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\epsilon \hat{H}_{\mathrm{eff}}}]^N$ with $\epsilon = \frac{\Delta t}{N} = \frac{t_f - t_i}{N}$. Using wavefunction representations for the position and momentum eigenstates associated to all extended phase space operators $(\hat{r}, \hat{p}_r)$, $(\hat{\theta}, \hat{p}_\theta)$, $(\hat{z}, \hat{p}_z)$, $(\hat{\xi}, \hat{p}_\xi)$ and $(\hat{\eta}^a, \hat{\mathcal{P}}_a)$, this procedure leads to a discretized representation of the extended phase space path integral associated to the relevant matrix element of the BRST invariant evolution operator in configuration space (further details can be found in [3]).

In particular, the ghost sector is represented in terms of Grassmann odd variables, whose integration then combines with that over the Lagrange multiplier sector (ξ, p_ξ) and may be completed exactly, to lead in effect to the gauge fixing choice implied by the differential equation (20) and the function $F(\xi)$. In the limit $N \to \infty$, as a consequence of the BRST invariance of the quantity being computed, the integration over that sector thus leads to an integral over Teichmüller space parametrized by the parameter γ and with the measure $\mathrm{d}\gamma/(2\pi\hbar)$, but over a domain $\mathcal{D}_\gamma[F]$ determined from the differential equation $\mathrm{d}\xi/\mathrm{d}t = F(\xi)$ and the boundary value $\xi_f = \xi(t_f)$ precisely in the manner discussed in section 2. Hence, even though the quantity of interested is BRST and gauge invariant, and thus defined over Teichmüller space rather than the space of Lagrange multiplier functions $\xi(t)$, nevertheless this gauge invariant quantity *is not independent of the gauge fixing procedure* [3, 15]. Indeed, as the examples of section 2 have illustrated, the domain $\mathcal{D}_\gamma[F]$ obtained through the classes of gauge fixings implied by the choice (20) is dependent both on the type of function $F(\xi)$ and on the parameters defining that function. Once again, gauge invariance of physical observables is not all there is to gauge invariant systems.

The integration over the radial variable r is precisely that which appears in the path integral calculation of the propagator of the two-dimensional spherically symmetric harmonic oscillator. Using the techniques developed in [17], one then finally obtains

$$\begin{aligned} P^{[F]}_{\mathrm{BRST}}(i \to f) = \mathrm{i} \sum_{\ell=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\ell(\theta_f - \theta_i)} \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi\hbar} \mathrm{e}^{\mathrm{i}p(z_f - z_i)/\hbar} \mathrm{e}^{-\mathrm{i}p^2 \Delta t/(2\hbar)} \\ \times \int_{\mathcal{D}_\gamma[F]} \frac{\mathrm{d}\gamma}{2\pi\hbar} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\gamma(p+g\ell)} \frac{\omega}{2\mathrm{i}\pi\hbar \sin\omega\Delta t} \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}|\ell|} \mathrm{e}^{\frac{\mathrm{i}\omega}{2\hbar}\frac{\cos\omega\Delta t}{\sin\omega\Delta t}(r_f^2 + r_i^2)} J_{|\ell|}\left(\frac{\omega r_f r_i}{\hbar \sin\omega\Delta t}\right). \end{aligned} \tag{73}$$

In order to further reduce this latter result and compare it with the exact expression (59) obtained for the admissible gauge fixing $z(t) = 0$, it should be clear that a choice for the function $F(\xi)$ must be made such that the corresponding gauge fixing is also admissible, thereby leading to the domain $\mathcal{D}_\gamma[F]$ being the entire real line. For any nonadmissible choice for $F(\xi)$, the associated domain $\mathcal{D}_\gamma[F]$ would not be the entire Techmüller space of the system, and the final result obtained for the nevertheless gauge invariant quantity $P^{[F]}_{\mathrm{BRST}}(i \to f)$ would not represent the correct expression (59) for the physical propagator of gauge invariant states of the model†.

Thus, assuming now that the function $F(\xi)$ defines an *admissible* gauge fixing of the system‡—with the meaning defined in the introduction—it is clear that, finally, the exact evaluation of the exact path integral representation of the BRST invariant propagator leads to the expression

$$P^{[\text{admissible } F]}_{\mathrm{BRST}}(i \to f) = \frac{\mathrm{i}}{2\pi\hbar}\frac{\omega}{2\mathrm{i}\pi\hbar\sin\omega\Delta t}\mathrm{e}^{\frac{\mathrm{i}\omega}{2\hbar}\frac{\cos\omega\Delta t}{\sin\omega\Delta t}(r_f^2+r_i^2)} \times \sum_{\ell=-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}|\ell|}\mathrm{e}^{\mathrm{i}\ell(\varphi_f-\varphi_i)}\mathrm{e}^{-\frac{\mathrm{i}}{2}\hbar\Delta t g^2\ell^2}J_{|\ell|}\left(\frac{\omega r_f r_i}{\hbar\sin\omega\Delta t}\right) \tag{74}$$

in which we have used the definitions $\varphi_{i,f} = \theta_{i,f} - gz_{i,f}$ of the gauge invariant combinations of the boundary conditions, which indeed appear in this expression as initially expected for a gauge invariant quantity.

Quite clearly, except for the overall normalization factor $\mathrm{i}/(2\pi\hbar)$ stemming from the quantum dynamics of the extended sector of variables (ξ, p_ξ) and $(\eta^a, \mathcal{P}_a)$ in the BFV-BRST invariant framework, the final result in (74) *valid only for an admissible gauge fixing* coincides *exactly* with the exact result in (59) obtained within the reduced phase space approach based on the admissible gauge fixing condition $z(t) = 0$ for the same physical propagator of the quantized model. In particular, this conclusion provides an explicit demonstration of the well known fact that the extended sector of Lagrange multiplier and ghost degrees of freedom of opposite Grassmann parities precisely cancels out the contributions of the gauge variant states to the matrix elements of physical observables, while only those contributions of physical states are retained.

6. The physical projector

The previous two sections have demonstrated how through a rather involved process of gauge fixing requiring a careful analysis of the possible Gribov problems which may thereby ensue, the configuration space propagator of the physical states of the quantized system may be obtained, but with an expression which is physically correct *only for an admissible choice of gauge fixing* free of any local and global Gribov ambiguity. Both approaches rely first on Dirac's Hamiltonian formulation of constrained systems, which is then either reduced or extended before the quantum dynamics of the system is considered.

The purpose of this section is to illustrate that all these complications may be avoided altogether, without the necessity of any gauge fixing whatsoever, by working immediately within Dirac's quantization of constrained systems and exploiting the construction of the physical projector onto gauge invariant states introduced in [8]. In particular, we shall consider once again the calculation of the physical propagator for the choice of boundary conditions (8), to explicitly establish that the correct result (59) is indeed readily derived using such an approach. This analysis only requires the results of Dirac's quantization derived in section 3.

† For example, with the choice $F(\xi) = a\xi^3$, the final result for the BRST invariant propagator would depend on the parameter a defining the gauge fixing condition, and only in the limit $a \to 0$ would the admissible result be recovered. Nevertheless, by construction, $P^{[F]}_{\mathrm{BRST}}(i \to f)$ is gauge invariant whatever the value for a, since the BFV-BRST path integral is gauge invariant independently of the choice of $F(\xi)$.

‡ Such as, for example, $F(\xi) = a\xi + b$.

In that section, it was shown that the spectrum of the $U(1)$ gauge symmetry generator $\hat{\phi}$ is given by

$$\hat{\phi} : p + \hbar g\ell \qquad -\infty < p < +\infty \qquad \ell = 0, \pm 1, \pm 2, \ldots . \tag{75}$$

This spectrum being continuous—due to the noncompact component of the gauge symmetry group related to the translations in the variable z which it induces—the actual definition of the projection operator onto physical states annihilated by the operator $\hat{\phi}$ requires [8] one to consider a finite eigenvalue interval $[-\delta, \delta]$, with a positive quantity δ taken as small as desired. The projection operator onto $\hat{\phi}$ eigenstates whose eigenvalue lies within that interval is then given by [8]

$$\mathbb{E}_\delta = \mathbb{E}[-\delta < \hat{\phi} < \delta] = \int_{-\delta}^{+\delta} \mathrm{d}\delta_0 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\gamma}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\gamma(\hat{\phi}-\delta_0)} = \int_{-\infty}^{+\infty} \mathrm{d}\gamma \frac{\sin(\delta\gamma/\hbar)}{\pi\gamma} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\gamma\hat{\phi}}. \tag{76}$$

Considering then the quantum evolution operator (45) in Dirac's quantization, its projection onto contributions from physical states only is defined by

$$\hat{U}_{\mathrm{phys}}(t_f, t_i) = \lim_{\delta\to 0} \frac{\pi\hbar}{\delta} \hat{U}(t_f, t_i)\mathbb{E}_\delta = \lim_{\delta\to 0} \frac{\pi\hbar}{\delta} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Delta t \hat{H}} \mathbb{E}_\delta \tag{77}$$

where $\hat{H}$ is the first-class quantum Hamiltonian of the system. Note that since $\hat{H}$ and $\hat{\phi}$ commute, one may also write

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Delta t \hat{H}} \mathbb{E}_\delta = \mathbb{E}_\delta \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Delta t \mathbb{E}_\delta \hat{H} \mathbb{E}_\delta} \mathbb{E}_\delta \tag{78}$$

in order to emphasize the fact that indeed only physical states contribute to the physical evolution operator $\hat{U}_{\mathrm{phys}}(t_f, t_i)$, both as intermediate and as external states, in the limit $\delta \to 0$.

Hence, the configuration space physical propagator is simply given by the following matrix element†:

$$P_{\mathrm{proj}}(i \to f) = \langle r_f, \theta_f, z_f | \hat{U}_{\mathrm{phys}}(t_i, t_f) | r_i, \theta_i, z_i \rangle \tag{79}$$

whose explicit evaluation only requires the wavefunctions of a complete basis of all quantum states of the system—including the nonphysical ones—constructed in (33). Since this basis diagonalizes both the Hamiltonian $\hat{H}$ and the generator $\hat{\phi}$, the calculation is rather straightforward and follows much the same lines as the one in section 4 for the summation over the integer m associated to the (r, θ) sector of degrees of freedom. The additional contributions from the z sector are easily included, since the projector $\mathbb{E}_\delta$ implies the constraint $p + \hbar g\ell = 0$ through a $\delta(p + \hbar g\ell)$ function in the limit $\delta \to 0$.

To explicitly illustrate how the contributions of gauge variant states as intermediate states are indeed projected out from the physical propagator, let us consider the expression for (79) when the complete set of eigenstate configuration space wavefunctions (33) is substituted for,

$$\begin{aligned} P_{\mathrm{proj}}(i \to f) = \sum_{m=0}^{+\infty} \sum_{\ell=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}p \psi_{m,\ell,p}(r_f, \theta_f, z_f) \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Delta t[\frac{1}{2}p^2 + \hbar\omega(2m+|\ell|+1)]} \\ \times \int_{-\infty}^{+\infty} \mathrm{d}\gamma \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\gamma(p+\hbar g\ell)} \psi^*_{m,\ell,p}(r_i, \theta_i, z_i). \end{aligned} \tag{80}$$

In this form, it is clear that the integration over the Teichmüller parameter γ enforces the gauge invariance constraint through the factor $2\pi\hbar\delta(p + \hbar g\ell)$. From this contribution, the numerical factor $2\pi\hbar$ cancels precisely the factor $1/(2\pi\hbar)$ which stems from the normalization of the wavefunctions (33) in the (z, p_z) sector of degrees of freedom, while the $\delta(p + \hbar g\ell)$ function implies that only those states for which $p = -\hbar g\ell$, namely physical states, are retained in the

† The coherent state matrix elements of the same operator have been considered in [8].

summation over the integers m and ℓ and the integration over the conserved momentum p. Hence, the physical projector, leading to an integration over Teichmüller space, does indeed project out the contributions of the gauge variant states of the quantized system; only physical states contribute as intermediate states to the physical propagator, each and every one of them contributing with the same multiplicity of unity.

The remainder of the calculation is then identical to that which leads to the result (59) established in the reduced phase space approach. Indeed, the summation over the integers m and ℓ is that already performed in that case, with an identical normalization of the wavefunctions in the (r, θ) sector of degrees of freedom. Hence, a direct enough and exact calculation readily provides the following final expression for the physical propagator within the physical projector approach:

$$P_{\text{proj}}(i \to f) = \frac{\omega}{2\mathrm{i}\pi\hbar \sin\omega\Delta t} \mathrm{e}^{\frac{\mathrm{i}\omega}{2\hbar}\frac{\cos\omega\Delta t}{\sin\omega\Delta t}(r_f^2+r_i^2)} \times \sum_{\ell=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}|\ell|}\mathrm{e}^{\mathrm{i}\ell(\varphi_f-\varphi_i)}\mathrm{e}^{-\frac{\mathrm{i}}{2}\hbar\Delta t g^2\ell^2} J_{|\ell|}\left(\frac{\omega r_f r_i}{\hbar\sin\omega\Delta t}\right) \tag{81}$$

a result which coincides *exactly* with those in (59) and (74) established on the basis of some gauge fixing procedure whose admissibility must be ascertained with great care, and by using methods going beyond the simpler framework of Dirac's quantization of constrained systems.

As it should, the physical projector has thus achieved the required projecting out of the contributions of the gauge variant states to the physical propagator, independently of any gauge fixing procedure and thereby avoiding any potential Gribov problem. With hindsight, the result (81) could have been obtained straightforwardly in section 3 within Dirac's quantization of the system [11], by restricting by hand (as one might say) the summation in (80) over the intermediate states contributing to the evolution operator (45) to the subspace of the physical states only, namely

$$\sum_{m=0}^{+\infty}\sum_{\ell=-\infty}^{+\infty} \psi_{m,\ell,p}(r_f,\theta_f,z_f)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Delta t[\frac{1}{2}\hbar^2g^2\ell^2+\hbar\omega(2m+|\ell|+1)]}\psi^*_{m,\ell,p}(r_i,\theta_i,z_i) \tag{82}$$

a relation in which the energy spectrum of gauge invariant states is also accounted for. Apart from an overall normalization factor of $1/(2\pi\hbar)$ stemming from the (z, p_z) sector of degrees of freedom, quite obviously the same result as established above for the physical propagator is obtained. Nevertheless, for systems not as simple as the present one, such a direct restriction to physical state contributions only is not expected to be feasable 'by hand' in such a straightforward way within Dirac's quantization, while the physical projector which finds its rightful setting within the same framework is, in general, the appropriate tool for that purpose—which is then achieved directly, albeit often implicitly.

7. Conclusions

Using the simple, yet rich enough, solvable $U(1)$ gauge invariant quantum mechanical model of [6], this work has demonstrated the clear advantages of using the physical projector [8] in the quantization of gauge invariant systems. Indeed, by its very definition, the physical projector avoids the apparent necessity of some gauge fixing procedure, which most often is at the origin of local and global Gribov problems rendering the associated gauge invariant description of a given system physically in- or over-complete. Moreover, all methods of gauge fixing require an approach going beyond the original Dirac–Hamiltonian framework for constrained systems, within which, however, the physical projector finds its rightful setting.

As this paper has clearly illustrated, the physical projector naturally avoids all the complications inherent in any gauge fixing procedure, both through the automatic lack of any Gribov problem and by the absence of any further developments beyond those required in any case by Dirac's approach. In addition, the physical projector implicitly induces the correct integration of all gauge orbits of a gauge invariant system [10] by effectively accounting for the contributions to the dynamics of the system, be it classical or quantum, of each one of all the gauge orbits once and only once—the very fact which characterizes precisely what should define an admissible gauge fixing procedure when one is being implemented.

In particular, by considering the explicit calculation of the physical propagator, it was clearly demonstrated how the physical projector automatically enforces the fact that only gauge invariant physical states contribute to that observable as intermediate states, the contributions of the gauge variant states being simply projected out. Within the other approaches to the quantization of gauge invariant systems, which all require some gauge fixing procedure and are thus potentially plagued by Gribov problems, the cancellation of the gauge variant contributions is achieved either by having only gauge invariant configurations survive the phase space reduction, or by having an extended sector of ghost degrees of freedom to compensate for the contributions of gauge variant configurations. It is only for an admissible gauge fixing that this cancellation of gauge variant contributions is correctly achieved—a feature which is directly, though implicitly, realized within the physical projector approach given the very character of the latter operator.

Quite obviously, the diverse advantages of the physical projector in a gauge fixing free quantization of gauge invariant systems are there to be explored much further for systems whose dynamics is not as simple as that of the model used in this paper, beginning, for example, with Yang–Mills theory in 1 + 1 and 2 + 1 dimensions, or topological quantum field theories [18]. And beyond such applications, it may be hoped that this approach to the quantization of gauge invariant systems will provide new insights [19] into the mathematical and physical riches of the actual gauge theories of the fundamental interactions.

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